

# A SIMPLE PROOF THAT THE POWER $\frac{2m}{m+1}$ IN THE BOHNENBLUST–HILLE INEQUALITIES IS SHARP

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ABSTRACT. The power  $\frac{2m}{m+1}$  in the polynomial (and multilinear) Bohnenblust–Hille inequality is optimal. This result is well-known but its proof highly nontrivial. In this note we present a quite simple proof of this fact.

## 1. INTRODUCTION

The polynomial and multilinear Bohnenblust–Hille inequalities were proved by H.F. Bohnenblust and E. Hille in 1931 and play a crucial role in different fields as Fourier and Harmonic Analysis and Quantum Information Theory (see [4, 5, 7]).

The polynomial Bohnenblust–Hille inequality proves the existence of a positive function  $C : \mathbb{N} \rightarrow [1, \infty)$  such that for every  $m$ -homogeneous polynomial  $P$  on  $\mathbb{C}^N$ , the  $\ell_{\frac{2m}{m+1}}$ -norm of the set of coefficients of  $P$  is bounded above by  $C_m$  times the supremum norm of  $P$  on the unit polydisc. This result has important striking applications in different contexts (see [4]). The multilinear version of the Bohnenblust–Hille inequality asserts that for every positive integer  $m \geq 1$  there exists a sequence of positive scalars  $(C_m)_{m=1}^\infty$  in  $[1, \infty)$  such that

$$\left( \sum_{i_1, \dots, i_m=1}^N |T(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \sup_{z_1, \dots, z_m \in \mathbb{D}^N} |T(z_1, \dots, z_m)|$$

for all  $m$ -linear forms  $T : \mathbb{C}^N \times \dots \times \mathbb{C}^N \rightarrow \mathbb{C}$  and every positive integer  $N$ , where  $(e_i)_{i=1}^N$  denotes the canonical basis of  $\mathbb{C}^N$  and  $\mathbb{D}^N$  represents the open unit polydisk in  $\mathbb{C}^N$ .

The original proof ([3]) that the power  $\frac{2m}{m+1}$  is optimal is quite puzzling. According to Defant *et al* ([4, page 486]), Bohnenblust and Hille “showed, through a highly nontrivial argument, that the exponent  $\frac{2m}{m+1}$  cannot be improved” or according to Defant and Schwarting [6, page 90], Bohnenblust and Hille showed “with a sophisticated argument that the exponent  $\frac{2m}{m+1}$  is optimal”. In [2] there is an alternative proof for the case of multilinear mappings, but the arguments are also nontrivial, involving  $p$ -Sidon sets and sub-Gaussian systems. The main goal of this note is to present a quite elementary proof (which solves simultaneously the cases of polynomials and multilinear mappings) of the optimality of  $\frac{2m}{m+1}$ .

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2. THE NEW PROOF OF THE SHARPNESS OF  $\frac{2m}{m+1}$ 

We will show that the optimality of the power  $\frac{2m}{m+1}$  is a straightforward consequence of the following famous result known as Kahane-Salem-Zygmund inequality (see [8, Theorem 4, Chapter 6] or [1, page 21]):

**Theorem 2.1** (Kahane-Salem-Zygmund inequality). *Let  $m, n$  be positive integers. Then there are signs  $\varepsilon_\alpha = \pm 1$  so that the  $m$ -homogeneous polynomial*

$$P_{m,n} : \ell_\infty^n \rightarrow \mathbb{C}$$

given by

$$P_{m,n} = \sum_{|\alpha|=m} \varepsilon_\alpha z^\alpha$$

satisfies

$$\|P_{m,n}\| \leq C n^{(m+1)/2} \sqrt{\log m}$$

where  $C$  is an universal constant (it does not depend on  $n$  or  $m$ ).

**Theorem 2.2.** *The power  $\frac{2m}{m+1}$  in the Bohnenblust–Hille inequalities is sharp.*

*Proof.* Let  $m \geq 2$  be a fixed positive integer. For each  $n$ , let  $P_{m,n} : \ell_\infty^n \rightarrow \mathbb{C}$  be the  $m$ -homogeneous polynomial satisfying the Kahane-Salem-Zygmund inequality. For our goals it suffices to deal with the case  $n > m$ .

Let  $q < \frac{2m}{m+1}$ . Then a simple combinatorial calculation shows that

$$\left( \sum_{|\alpha|=m} |\varepsilon_\alpha|^q \right)^{1/q} = \left( p(n) + \frac{1}{m!} \prod_{k=0}^{m-1} (n-k) \right)^{\frac{1}{q}},$$

where  $p(n) > 0$  is a polynomial of degree  $m-1$ . If the polynomial Bohnenblust–Hille inequality was true with the power  $q$ , then there would exist a constant  $C_{m,q} > 0$  so that

$$C_{m,q} C \geq \frac{1}{n^{(m+1)/2} \sqrt{\log m}} \left( p(n) + \frac{1}{m!} \prod_{k=0}^{m-1} (n-k) \right)^{1/q}$$

for all  $n$ . If we raise both sides to the power of  $q$  and make  $n \rightarrow \infty$  we obtain

$$(C_{m,q} C)^q \geq \lim_{n \rightarrow \infty} \left( \frac{r(n)}{m! n^{q(m+1)/2} (\sqrt{\log m})^q} + \frac{p(n)}{n^{q(m+1)/2} (\sqrt{\log m})^q} \right),$$

with

$$r(n) = \prod_{k=0}^{m-1} (n-k).$$

Since

$$\deg r = m > \frac{q(m+1)}{2}$$

we have

$$\lim_{n \rightarrow \infty} \left( \frac{r(n)}{m! n^{q(m+1)/2} (\sqrt{\log m})^q} + \frac{p(n)}{n^{q(m+1)/2} (\sqrt{\log m})^q} \right) = \infty,$$

a contradiction. Since the multilinear Bohnenblust–Hille inequality (with a power  $q$ ) implies the polynomial Bohnenblust–Hille inequality with the same power, we conclude that  $\frac{2m}{m+1}$  is also sharp in the multilinear case.  $\square$

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